MULTILINEAR CALDERÓN-ZYGMUND OPERATORS ON HARDY SPACES, II

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ABSTRACT. In this note we explain a point left open in the literature of Hardy spaces, namely that for a sufficiently smooth m-linear Calderón-Zygmund operator bounded on a product of Lebesgue spaces we have

$$T(f_1,\dots,f_m)=\sum_{i_1}\cdots\sum_{i_m}\lambda_{1,i_1}\cdots\lambda_{m,i_m}T(a_{1,i_1},\dots,a_{m,i_m})\quad \text{a.e.},$$

where a_{j,i_j} are H^{p_j} atoms, $\lambda_{j,i_j}\in\mathbf{C}$, and $f_j=\sum_{i_j}\lambda_{j,i_j}a_{j,i_j}$ are H^{p_j} distributions. In some particular cases the proof is new even when $m=1$.

1. INTRODUCTION

This article is a subsequent article of [7] and is concerned with the boundedness of multilinear Calderón-Zygmund operators on products of Hardy spaces. Our goal in this article is to prove that the action of a sufficiently smooth m-linear Calderón-Zygmund operator can be interchanged with infinite sums of atoms; see identity (3). We also discuss approximations of general m-linear Calderón-Zygmund operators by sequences of smoother ones.

Multilinear Calderón-Zygmund operators were introduced by Coifman and Meyer [4], [5] but were not systematically studied for about a quarter century until the appearance of [9] and its subsequent article [8]. The boundedness of these operators on products of Hardy H^p spaces first appeared in [7]. Subsequently, the articles [10], [11] studied the boundedness of these operators from product Hardy spaces into Hardy spaces under some additional conditions; the related work in [3] focuses on singular integrals in product spaces.

We begin by giving a precise definition of Calderón-Zygmund operators. Let $\mathscr{D}(\mathbf{R}^n)$ be the space of smooth functions with compact support.

Definition 1.1. *An m-linear operator $T:\ \mathscr{D}(\mathbf{R}^n)\times\mathscr{D}(\mathbf{R}^n)\times\cdots\times\mathscr{D}(\mathbf{R}^n)\to\mathscr{D}'(\mathbf{R}^n)$, whose Schwartz kernel S coincides with a function K away from diagonal $\{(y_0,y_1,\dots,y_m): y_0=y_1=\cdots=y_m\}$ on $(\mathbf{R}^n)^{m+1}$, is called a Calderón-Zygmund operator if*

$$\left|\partial_{y_0}^{\alpha_0}\partial_{y_1}^{\alpha_1}\cdots\partial_{y_m}^{\alpha_m}K(y_0,y_1,\dots,y_m)\right|\le\frac{A_\alpha}{d(\vec{y})^{mn+|\alpha|}},\tag{1}$$

where $d(\vec{y}) = \sum_{i,l=0}^{m} |y_i - y_l|$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a multiindex with $|\alpha| \le I$ and I is some large integer; and there exist $1 < q_j < \infty$ and $1 \le q$ such that

$$\|T(f_1, f_2, \dots, f_m)\|_{L^q} \le A\,\|f_1\|_{L^{q_1}}\,\|f_2\|_{L^{q_2}} \cdots \|f_m\|_{L^{q_m}}, \tag{2}$$

where $\frac{1}{q} = \sum_{j=1}^{m} \frac{1}{q_j}$. We call A and A_α the associated constants of T.

If there is no confusion we denote $d(\vec{y})$ simply by d.

Remark 1.2. *The fact that S is a Schwartz kernel of T means that S is an element of $\mathscr{D}'(\mathbf{R}^{(m+1)n})$ and such that for all $f_j, g \in \mathscr{D}(\mathbf{R}^n)$ we have*

$$\langle T(f_1, \dots, f_m), g\rangle = \langle S, g \otimes f_1 \otimes \cdots \otimes f_m\rangle .$$

Here $(g \otimes f_1 \otimes \cdots \otimes f_m)(y_0, y_1, \dots, y_m) = g(y_0) f_1(y_1) \cdots f_m(y_m)$.

Remark 1.3. *If T maps $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 < q_j < \infty$, $1 \le q < \infty$, then T also maps $L^{r_1} \times \cdots \times L^{r_m}$ to L^r for any r_j, r with $1 < r_j < \infty$, $\frac{1}{r} = \sum_{j=1}^{m} \frac{1}{r_j}$; see* [9].

In [7] it was proved that T is bounded from $H^{p_1} \times \cdots \times H^{p_m}$ to L^p with $\frac{1}{p} = \sum_j \frac{1}{p_j}$ by showing that $\|T(a_1, \dots, a_m)\|_{L^p} \le C$, where a_j are H^{p_j}-atoms. The proof relies on identity (3) below, which was left unproved there. The proof of (3), although trivial for finite sums of atoms, is quite delicate and requires substantial work for infinite sums. The details of the argument are carefully described in this article. Note that it does not suffice to know the validity of (3) for finite sums of atoms, to derive the boundedness of T on $H^{p_1} \times \cdots \times H^{p_m}$. In fact, although the set $\mathscr{F}$ of finite combinations of atoms is dense in H^{p_j}, Bownik [2], inspired by an idea of Meyer (contained in [13]), constructed an example of a linear functional on a dense subspace of H^1 that is uniformly bounded on $\mathscr{F}$ but does not extend to a bounded linear functional on the whole H^1.

The main goal of this article is to provide a proof for the following result:

Theorem 1.4. *Let $0 < p_j \le 1 < q_j < \infty$ and let $f_j \in H^{p_j} \cap L^{q_j}$ have atomic decompositions $f_j = \sum_{i_j} \lambda_{j,i_j} a_{j,i_j}$, where $\lambda_{j,i_j} \in \mathbf{C}$, $\|f_j\|_{H^{p_j}}^{p_j} \approx \sum_{i_j} |\lambda_{j,i_j}|^{p_j}$, and a_{j,i_j} are L^∞ atoms for H^{p_j}. Let T be a Calderón-Zygmund operator as in Definition 1.1 which satisfies* (2). *Then for almost all $x \in \mathbf{R}^n$ we have*

$$T_k(f_1, \dots, f_m)(x) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \lambda_{1,i_1} \dots \lambda_{m,i_m} T_k(a_{1,i_1}, \dots, a_{m,i_m})(x), \tag{3}$$

and consequently, T maps $H^{p_1} \times \cdots \times H^{p_m}$ to L^p, when $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

In Sections 2 and 3 we prove Theorem 1.4 while in Sections 4 and 5 we provide an alternative proof of identity (3) for Calderón-Zygmund operators defined as almost pointwise limits of smoother operators.

2. THE APPROXIMATIONS $\{T_k\}$

To obtain the proof of (3) we need to introduce a new operator T_k which is defined as

$$T_k(f_1,\dots,f_m) = R_kT(R_kf_1,\dots,R_kf_m),$$

where $R_k(f) = \phi_k * f$, ϕ is a nonnegative smooth radial function supported in $B(0,1)$ (the unit ball of radius 1) whose integral is 1, and $\phi_k(x) = k^n\phi(kx)$. Denote by S_k the kernel of T_k. For this type of approximations in the linear case, we refer to [12] and [1].

Since $\|R_k(f)\|_{L^q} \le \|f\|_{L^q}$ and $\|R_kf - f\|_{L^q} \to 0$ as $k \to \infty$ for $q \ge 1$, it's easy to check that $T_k(f_1,\dots,f_m) \to T(f_1,\dots,f_m)$ in L^q whenever $f_j \in L^{q_j}$, $\frac{1}{q} = \sum_{j=1}^m \frac{1}{q_j}$, $\infty > q_j > 1$ and $q \ge 1$. These T_k's satisfy several nice properties which we state in the next two results.

Proposition 2.1. *$\{T_k\}$ is a collection of Calderón-Zygmund operators with equivalent constants.*

Proof. We observe that $\|T_k(f_1,\dots,f_m)\|_{L^q} \le C\|f_1\|_{L^{q_1}}\cdots\|f_m\|_{L^{q_m}}$, since $\|R_kf\|_{L^q} \le \|f\|_{L^q}\|\phi\|_{L^1}$, thus one requirement in Definition 1.1 is satisfied.

Next, we check condition (1) on the kernel S_k of T_k

$$\left|\partial_{y_0}^{\alpha_0}\partial_{y_1}^{\alpha_1}\dots\partial_{y_m}^{\alpha_m}S_k(y_0,y_1,\dots,y_m)\right| \le \frac{A_\alpha}{d(\vec{y})^{mn+|\alpha|}},$$

which, by an easy calculation, is defined at every point by

$$S_k(y_0,y_1,\dots,y_m) = \langle T(\tau_{y_1}\phi_k,\dots,\tau_{y_m}\phi_k),\tau_{y_0}\phi_k\rangle,$$

where $\tau_y f(x) = f(x-y)$.

We consider two cases concerning the size of $d(\vec{y})$ when we fix k.

Case 1. $d(\vec{y}) > \frac{4r}{k}$, where $r = C_{m+1}^2 + 1$ and $C_{m+1}^2 = \frac{(m+1)m}{2}$. If we choose the largest term $|y_{l_1} - y_{l_2}|$ among $|y_i - y_l|$, we have $|y_{l_1} - y_{l_2}| > \frac{4}{k}$ and therefore $supp\ \tau_{y_{l_1}}\phi_k \cap supp\ \tau_{y_{l_2}}\phi_k = \emptyset$. As a result, $S_k(y_0,\dots,y_m)$ can be written as

$$\int_{\mathbf{R}^{(m+1)n}} K(y_0 - \tfrac{u_0}{k}, y_1 - \tfrac{u_1}{k},\dots,y_m - \tfrac{u_m}{k})\phi(u_0)\phi(u_1)\dots\phi(u_m)du_0du_1\dots du_m.$$

Since $|(y_{l_1} - \frac{u_{l_1}}{k}) - (y_{l_2} - \frac{u_{l_2}}{k})| > \frac{1}{2}|y_{l_1} - y_{l_2}|$ for $u_{l_i} \in supp\ \phi$, $i = 1,2$,

$$\begin{aligned}|\partial_{y_0}^{\alpha_0}\partial_{y_1}^{\alpha_1}\dots\partial_{y_m}^{\alpha_m}S_k(y_0,y_1,\dots,y_m)| \le& \frac{A_\alpha}{\left(\sum_{i,l}|(y_i - \frac{u_i}{k}) - (y_l - \frac{u_l}{k})|\right)^{mn+|\alpha|}}\\ \le& \frac{A_\alpha}{(\frac{1}{2}|y_{l_1} - y_{l_2}|)^{mn+|\alpha|}}\\ \le& \frac{CA_\alpha}{d(\vec{y})^{mn+|\alpha|}}\end{aligned}$$

where the constant C is independent of k.

Case 2. $d(\vec{y}) \leq \frac{4r}{k}$. We can use the boundedness of T to get

$$\begin{aligned}
&|\partial_{y_0}^{\alpha_0}\partial_{y_1}^{\alpha_1}\cdots\partial_{y_m}^{\alpha_m}S_k(y_0,y_1,\dots,y_m)|\\
=&|\langle T(\tau_{y_1}(\partial^{\alpha_1}\phi_k),\dots,\tau_{y_m}(\partial^{\alpha_m}\phi_k)),\tau_{y_0}(\partial^{\alpha_0}\phi_k)\rangle|\\
\leq& Ak^{(m+1)n}k^{|\alpha|}k^{-\frac{n}{q_1}}\cdots k^{-\frac{n}{q_m}}k^{-\frac{n}{q'}}\|\phi\|_{L^{q_1}}\cdots\|\phi\|_{L^{q_m}}\|\phi\|_{L^{q'}}\\
=&Ck^{mn+|\alpha|}\\
\leq& Cd(\vec{y})^{-(mn+|\alpha|)},
\end{aligned}$$

where again C is independent of k. □

Note that the preceding proof shows that each function $\partial_{y_0}^{\alpha_0}\partial_{y_1}^{\alpha_1}\cdots\partial_{y_m}^{\alpha_m}S_k$ is bounded by $Ck^{mn+|\alpha|}$.

Proposition 2.2. *For fixed positive integers k and J, operators with kernels of the form $\partial^\beta S_k$, where $|\beta| \leq J$, form a collection of Calderón-Zygmund operators with the same associated constants, which depend only on J.*

Proof. Any finite collection of Calderón-Zygmund operators can be made to have the same associated constants, so we need only to check that each T_β with kernel $\partial^\beta S_k$ is a Calderón-Zygmund operator.

We have proved the case $\beta = 0$ in Proposition 2.1. For the case $\beta \neq 0$, if $d > \frac{4r}{k}$, where r is as in Proposition 2.1, then $|\partial^\alpha\partial^\beta S_k| \leq Cd^{-mn-|\alpha|-|\beta|} \leq Ck^{|\beta|}d^{-mn-|\alpha|}$. If $d \leq \frac{4r}{k}$ and $\beta \neq 0$, then $|\partial^\alpha\partial^\beta S_k| \leq Ck^{mn+|\beta|+|\alpha|} \leq Ck^{|\beta|}d^{-mn-|\alpha|}$. We have proved that T_β satisfies (1) in Definition 1.1.

Concerning the boundedness of T_β on some product of Lebesgue spaces, we take $q = 1$ and use the property that $\partial^\beta S_k$ is bounded to get

$$\begin{aligned}
&\|T_\beta(f_1,f_2,\dots,f_m)\|_{L^1}\\
\leq&\int\|\partial^\beta S_k(\cdot,y_1,y_2,\dots,y_m)\|_{L^1}|f_1(y_1)f_2(y_2)\cdots f_m(y_m)|dy_1dy_2\cdots dy_m\\
\leq& C\int|f_1(y_1)f_2(y_2)\cdots f_m(y_m)|dy_1dy_2\cdots dy_m\\
\leq& C\|f_1\|_{L^{q_1}}\|f_2\|_{L^{q_2}}\cdots\|f_m\|_{L^{q_m}},
\end{aligned}$$

provided $1 = \sum_{j=1}^m \frac{1}{q_j}$ and $\|\partial^\beta S_k(\cdot,y_1,y_2,\dots,y_m)\|_{L^1} \leq C < \infty$. To verify the latter, we split the integral into two parts

$$\begin{aligned}
&\Big(\int_{d\leq\frac{4r}{k}}+\int_{d>\frac{4r}{k}}\Big)|\partial^\beta S_k(x,y_1,y_2,\dots,y_m)|dx\\
\leq&\int_{d\leq\frac{4r}{k}}k^{mn+|\beta|}dx+\int_{d>\frac{4r}{k}}d^{-mn-|\beta|}dx := \mathrm{I}+\mathrm{II}
\end{aligned}$$

Obviously $|\mathrm{I}| \le \int_{|x-y_1|\le \frac{4r}{k}} k^{mn+|\beta|} dx \le C_k$.

When $d > \frac{4r}{k}$, if $\sum_{i,l=1}^m |y_i - y_l| > \frac{2r}{k}$, then

$$|\mathrm{II}| \le \int_{\mathbf{R}^n} \frac{1}{(\frac{2r}{km} + |x-y_i|)^{mn+|\beta|}} dx < \infty$$

since $|\beta| \ge 1$ and $m \ge 1$. If $\sum_{i,l} |y_i - y_l| < \frac{2r}{k}$, then $\sum_{i=1}^m |x-y_i| > \frac{2r}{k}$ and we must have l such that $|x-y_l| > \frac{2r}{km}$ and then let us choose $i \ne l$, as a result

$$|\mathrm{II}| \le \int_{\mathbf{R}^n} \frac{1}{(\frac{2r}{km} + |x-y_i|)^{mn+|\beta|}} dx < \infty.$$

By now we have checked the boundedness requirement in Definition 1.1 and therefore we conclude the proof. $\square$

We now study the behavior of T_k on Hardy spaces. Let us fix p_j with $0 < p_j \le 1$. Since T_k are Calderón-Zygmund operators with equivalent constants as T, by the results in [7], there is a constant C independent of k such that $\|T_k(a_1, \dots, a_m)\|_{L^p} \le C$ if a_j is an H^{p_j}-atom, $j = 1, \dots, m$. In the next section we show that for all $x \in \mathbf{R}^n$ we have

$$T_k(f_1, \dots, f_m)(x) = \sum_{i_1=1}^\infty \cdots \sum_{i_m=1}^\infty \lambda_{1,i_1} \cdots \lambda_{m,i_m} T_k(a_{1,i_1}, \dots, a_{m,i_m})(x) \tag{4}$$

whenever $f_j \in H^{p_j} \cap L^{q_j}$ and $f_j = \sum_{i_j=1}^\infty \lambda_{j,i_j} a_{j,i_j}$ in $H^{p_j} \subset \mathscr{D}'$, which is arbitrary atomic decomposition of f_j such that $(\sum_{i_j} |\lambda_{j,i_j}|^{p_j})^{1/p_j} \le 2\|f_j\|_{H^{p_j}}$. Under these assumptions, we have

$$\begin{aligned}
\|T_k(f_1, \dots, f_m)\|_{L^p}^p &= \Big\| \sum_{i_1=1}^\infty \cdots \sum_{i_m=1}^\infty \lambda_{1,i_1} \cdots \lambda_{m,i_m} T_k(a_{1,i_1}, \dots, a_{m,i_m}) \Big\|_{L^p}^p \\
&\le C \sum |\lambda_{1,i_1} \cdots \lambda_{m,i_m}|^p \|T_k(a_{1,i_1}, \dots, a_{m,i_m})\|_{L^p}^p \\
&\le C \sum |\lambda_{1,i_1} \cdots \lambda_{m,i_m}|^p \\
&\le C (\sum |\lambda_{1,i_1}|^{p_1})^{\frac{p}{p_1}} \cdots (\sum |\lambda_{m,i_m}|^{p_m})^{\frac{p}{p_m}} \\
&\le C \|f_1\|_{H^{p_1}}^p \cdots \|f_m\|_{H^{p_m}}^p .
\end{aligned}$$

Since $H^{p_j} \cap L^{q_j}$ is dense in H^{p_j}, we can therefore extend T_k continuously on $H^{p_1} \times \cdots \times H^{p_m}$.

If we fix $f_j \in H^{p_j} \cap L^{q_j}$, then we can extract a subsequence k_i, which depends on $\{f_j\}_{j=1}^m$, such that $T_{k_i}(f_1, \dots, f_m)(x) \to T(f_1, \dots, f_m)(x)$ a.e.. Applying Fatou's lemma and $\|T_k(f_1, \dots, f_m)\|_{L^p}^p \le C\|f_1\|_{H^{p_1}}^p \cdots \|f_m\|_{H^{p_m}}^p$ with C independent of k, then we have

$$\|T(f_1, \dots, f_m)\|_{L^p} \le C\|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}$$

for all $f_j \in H^{p_j} \cap L^{q_j}$. Then T can be continuously extended on the entire $H^{p_1} \times \cdots \times H^{p_m}$ due to the density of $H^{p_j} \cap L^{q_j}$ in H^{p_j}.

3. $\partial^\beta S_k$ AS A CONTINUOUS LINEAR FUNCTIONAL ON H^{p_j}

In this section we establish the validity of (4). We begin with the following result.

Theorem 3.1. *Let $0 < p_i \le 1 < q_i < \infty$, $i \in \{1, \dots, m\}$ and fix k, x, j and $f_t \in H^{p_t} \cap L^{q_t}$ for $t \neq j$. Then the function*

$$y_j \mapsto \int_{\mathbf{R}^{(m-1)n}} S_k(x, y_1, \dots, y_m) \prod_{t \neq j} f_t(y_t) dy_1 \cdots dy_{j-1} dy_{j+1} \dots dy_m,$$

is a continuous linear functional on H^{p_j} if the I in Definition 1.1 satisfies $I \ge m + \sum_{j=1}^m [\alpha_j]$, where $\alpha_j = \frac{n}{p_j} - n$.

Equality (4) is a direct consequence of this theorem. Indeed, we take $I = m + \sum_{j=1}^m [\alpha_j]$, where I is as in Definition 1.1. If $f_1 = \sum_{i_1} \lambda_{1,i_1} a_{1,i_1}$ in $H^{p_1} \cap L^{q_1}$ and x is fixed in $\mathbf{R}^n$, then

$$\begin{aligned}
&T_k(f_1, \dots, f_m)(x) \\
&= \int_{\mathbf{R}^n} \int_{\mathbf{R}^{n(m-1)}} S_k(x, y_1, y_2, \dots, y_m) \prod_{i=2}^m f_i(y_i)\, dy_2 \cdots dy_m f_1(y_1) dy_1 \\
&= \sum_{i_1=1}^\infty \lambda_{1,i_1} \int_{\mathbf{R}^n} \int_{\mathbf{R}^{n(m-1)}} S_k(x, y_1, y_2, \dots, y_m) \prod_{i=2}^m f_i(y_i)\, dy_2 \cdots dy_m a_{1,i_1}(y_1) dy_1 \\
&= \sum_{i_1=1}^\infty \lambda_{1,i_1} T(a_{1,i_1}, f_2, \dots, f_m)(x),
\end{aligned}$$

since $\int S_k(x, y_1, y_2, \dots, y_m) \prod_{i=2}^m f_i(y_i)\, dy_2 \cdots dy_m \in (H^{p_1})^*$. Now use this idea iteratively with f_1 replaced by a_{1,i_1} to obtain (4).

A function g is in $L^q_\alpha(\mathbf{R}^n)$ if $g \in L^q_{loc}(\mathbf{R}^n)$ and there is a constant C such that for any cube $Q \subset \mathbf{R}^n$, there is a polynomial P of degree less than $[\alpha]$ such that

$$\left(\frac{1}{|Q|} \int_Q |g(x) - P(x)|^q dx \right)^{\frac{1}{q}} \le C|Q|^{\frac{\alpha}{n}}.$$

The smallest C such that the previous inequality is true is denoted by $\|g\|_{L^q_\alpha}$. This norm makes $L^q_\alpha(\mathbf{R}^n)$ a normed space if we identify functions whose difference is a polynomial of degree less than $[\alpha]$. We need a characterization of $(H^p)^*$ which is discussed in detail in [6].

Theorem A. *If $0 < p \le 1$, $\alpha = \frac{n}{p} - n$, $1 \le q \le \infty$ if $p < 1$ and $1 \le q < \infty$ if $p = 1$, then $(H^p(\mathbf{R}^n))^* = L^q_\alpha(\mathbf{R}^n)$.*

Proof of Theorem 3.1. By Theorem A, we need only to check that the function in Theorem 3.1 is in $L^{\infty}_{\alpha_j}(\mathbf{R}^n)$, which is exactly $(H^{p_j})^*$ if $0<p<1$ and a subspace of $(H^1)^*$ when $p_j=1$, since

$$\frac{1}{|Q|}\int_Q |f(x)|\,dx \le \|f\chi_Q\|_{L^\infty}.$$

We only consider the case $j=1$ here since the remaining cases are obtained by symmetry.

Let us consider the function $G_\beta(y_m)=\partial^\beta S_k(x,y_1,\dots,y_{m-1},y_m)$, where $x,\ y_1,\dots,\ y_{m-1}$ are fixed and $|\beta|\le J-([\alpha_m]+1)$ with J a fixed positive integer, which is larger than $m+\sum_{j=1}^m[\alpha_j]$ and we use α_j to denote $\frac{n}{p_j}-n$. We claim that $\|G_\beta\|_{L^q_{\alpha_m}}\le C$ with C independent of $x,y_1,\dots,y_{m-1}$ and β.

Indeed for any cube $Q=Q(x_0,r)$, a cube centered at x_0 with length r, there is a polynomial P such that $|G_\beta(y_m)-P(y_m)|\le C|Q|^{\frac{\alpha_m}{n}}$ for all y_m in Q. The functions $G_\beta=\partial^\beta S_k$ are bounded with bound C independent of $x,y_1,\dots,y_{m-1}$ and β, so we can take $P=0$ and show that $|G_\beta(y_m)|\le C\le C|Q|^{\alpha_m/n}$ whenever $r\ge 1$. Now we can restrict ourselves to the case where $r<1$. If we take P to be the Taylor polynomial of $G_\beta(y_m)$ at x_0 of degree $[\alpha_m]$, then

$$\begin{aligned}|G_\beta(y_m)-P(y_m)| &= \Big|\sum_{|\gamma|=[\alpha_m]+1}\partial^\gamma_{y_m}\partial^\beta S_k(x,y_1,\dots,y_{m-1},\xi)(y_m-x_0)^\gamma\Big|\\ &\le Ck^{mn+|\beta|+|\gamma|}r^{[\alpha_m]+1}\\ &\le Ck^{mn+|\beta|+|\gamma|}r^{\alpha_m},\end{aligned}$$

where C depends on the constants of the Calderón-Zygmund operators $T_{\gamma+\beta}$ with kernels $\partial^{\beta+\gamma}S_k$, but it's independent of k and r.

Now we have proved that $G_\beta\in(H^{p_m})^*$ with bounded norms for $|\beta|\le J-([\alpha_m]+1)$, as a result $\int G_\beta(y_m)f_m(y_m)dy_m=\sum\lambda_i\int G_\beta(y_m)a_i(y_m)dy_m$ if $f_m=\sum\lambda_i a_i$ is an atomic decomposition of f_m. Furthermore we can show that $|\langle G_\beta,f_m\rangle|\le C\|f_m\|_{H^{p_m}}$ with C independent of the variables again.

We will finish the proof by induction. Let us assume we have proved that, $\langle\partial^\beta S_k,f_{\nu+1}\otimes\cdots\otimes f_m\rangle$ with $|\beta|\le J-\sum_{i=\nu}^m([\alpha_i]+1)$, as functions of y_ν, are functions in $(H^{p_\nu})^*$ with norms bounded by

$$C\prod_{i=\nu+1}^m\|f_i\|_{H^{p_i}}, \tag{5}$$

where C is independent of $x,y_1,\dots,y_\nu$ and β. Fix f_i for $i\ge\nu$ and define

$$F_\beta(y_{\nu-1})=\langle\partial^\beta S_k,f_\nu\otimes\cdots\otimes f_m\rangle.$$

Next we show that $F_\beta(y_{\nu-1}) \in L^\infty_{\alpha_{\nu-1}} = (H^{p_{\nu-1}})^*$ with operator norm bounded by $C\prod_{i=\nu}^m \|f_i\|_{H^{p_i}}$. Again we only consider the case $r<1$ because of (5).

Take P as the Taylor polynomial of $F_\beta(y_{\nu-1})$ at x_0 of degree $[\alpha_{\nu-1}]$, then

$$\begin{aligned}
&|F_\beta(y_{\nu-1}) - P(y_{\nu-1})|\\
=&\Big|\sum_{|\gamma|=[\alpha_{\nu-1}]+1} \langle \partial^\gamma_{y_{\nu-1}} \partial^\beta S_k(x,\dots,y_{\nu-2},\xi,y_\nu,\dots), f_\nu \otimes \cdots \otimes f_m\rangle (y_{\nu-1}-x_0)^\gamma\Big|\\
\le& C\prod_{i=\nu}^m \|f_i\|_{H^{p_i}} r^{\alpha_{\nu-1}}
\end{aligned}$$

where C is independent of $x, y_1, \dots, y_{\nu-2}$ and β.

To summarize, we have proved that $\int \partial^\beta S_k \prod_{i\ge 2} f_i(y_i)dy_2\cdots dy_m$ is in $(H^{p_1})^*$ as a function of y_1 for $|\beta| \le J-(m+\sum_{j=1}^m[\alpha_j])$. We therefore obtain the conclusion of this theorem by symmetry. □

4. An alternative approach

We can prove a stronger result, namely that equality (4) is true if we replace T_k by T, where T is a Calderón-Zygmund operator defined in Definition 1.1 with the additional condition (6) below. A direct corollary of this result is that for such T we have

$$\|T(f_1,\dots,f_m)\|^p_{L^p} \le C\|f_1\|^p_{H^{p_1}} \cdots \|f_m\|^p_{H^{p_m}},$$

and the proof of this is the same as that for T_k given at the end of Section 2.

Pick a function $\Phi(x)$ which is $\mathscr{C}^\infty$ on $\mathbf{R}^n$ and is equal to 1 for $|x|\ge 2$ and vanishes for $|x|\le 1$. Then we define for $\varepsilon < 1/10$

$$K^{(\varepsilon)}(y_0,y_1,\dots,y_m) = K(y_0,y_1,\dots,y_m)[\Phi(d(\vec{y})/\varepsilon) - \Phi(\varepsilon d(\vec{y}))]$$

and it's easy to check that $|\partial^\alpha K^{(\varepsilon)}(\vec{y})| \le A_\alpha\, d(\vec{y})^{-mn-|\alpha|}$ uniformly in ε by considering $d(\vec{y})$ to be comparable to ε, $1/\varepsilon$ and otherwise respectively. We can define a truncated operator

$$T^{(\varepsilon)}(f_1,\dots,f_m)(y_0) = \int_{\mathbf{R}^{mn}} K^{(\varepsilon)}(y_0,y_1,\dots,y_m) f_1(y_1)\cdots f_m(y_m)\, dy_1\cdots dy_m.$$

If there exists a sequence $\{\varepsilon_k\}$ that tends to zero and we can define T as

$$T(f_1,\dots,f_m)(y_0) = \lim_{\varepsilon_k\to 0} T^{(\varepsilon_k)}(f_1,\dots,f_m)(y_0) \tag{6}$$

initially for Schwartz functions f_j. Examples of operators of this kind can be found in [9]. Actually if

$$\left|\int_{R_1<|y_1|+\cdots+|y_m|<R_2} K(y_1,\dots,y_m)\, dy_1\cdots dy_m\right| \le A < \infty \tag{7}$$

for all $0 < R_1 < R_2 < \infty$, then we can define an m-linear translation-invariant Calderón-Zygmund operator satisfying (6). By Cotlar's inequality in [8] we have that $T^{(\varepsilon)}$ maps $L^{q_1} \times \cdots \times L^{q_m}$ to L^q uniformly in ε, (6) is valid almost everywhere for any $f_j \in L^{q_j}$ and $T^{(\varepsilon_k)}(f_1, \ldots, f_m) \to T(f_1, \ldots, f_m)$ in L^q. Although the maximal operator in [8] is defined in a non-smooth way, while here we defined the maximal operator via smooth truncations, we still are able to apply results in [8] because the difference of these two operators is controlled by the product of the Hardy-Littlewood maximal operators.

Theorem 4.1. *Let T be an m-linear Calderón-Zygmund operator satisfying (6), then*

$$T(f_1, \ldots, f_m)(x) = \sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T(a_{1,i_1}, \ldots, a_{m,i_m})(x) \qquad \text{a.e.},$$

where $f_j = \sum_{i_j} \lambda_{j,i_j} a_{j,i_j}$ lies in $H^{p_j} \cap L^2$ and the sum provides an arbitrary atomic decomposition of f_j.

Proof. To prove this theorem, we first prove the analogous result for $T^{(\varepsilon)}$, i.e.,

$$T^{(\varepsilon)}(f_1, \ldots, f_m)(x) = \sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T^{(\varepsilon)}(a_{1,i_1}, \ldots, a_{m,i_m})(x) \ a.e..$$

Let us fix $\varepsilon > 0$. Then we have

$$\begin{aligned}
&\Big|\big\{\big|T^{(\varepsilon)}(f_1, \ldots, f_m) - \sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T^{(\varepsilon)}(a_{1,i_1}, \ldots, a_{m,i_m})\big| > \delta\big\}\Big| \\
\leq &\Big|\big\{\big|T^{(\varepsilon)}(f_1, \ldots, f_m) - \sum_{\max(i_1,\ldots,i_m) \leq L} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T^{(\varepsilon)}(a_{1,i_1}, \ldots, a_{m,i_m})\big| > \frac{\delta}{2}\big\}\Big| \\
&+ \Big|\big\{\big|\sum_{\max(i_1,\ldots,i_m) > L} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T^{(\varepsilon)}(a_{1,i_1}, \ldots, a_{m,i_m})\big| > \frac{\delta}{2}\big\}\Big| \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

The kernels $K^{(\varepsilon)}$ satisfy the same assumptions as K uniformly in ε, therefore in view of the results in [7] we have that

$$\mathrm{II} \leq C(\tfrac{2}{\delta})^p \Big(\sum_{j=1}^{m} \big(\sum_{k_j = L}^{\infty} |\lambda_{j,k_j}|^{p_j} \big)^{1/p_j} \prod_{r \neq j} \|f_r\|_{H^{p_r}} \Big)^p,$$

which tends to 0 as $L \to \infty$.

To estimate term I, we need the following lemma which will be proved in next section.

Lemma 4.2. *If $f_j \in L^2 \cap H^{p_j}$, then*

$$\left| \int_{\mathbf{R}^{mn}} K^{(\varepsilon)}(y_0, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m \right| \leq C_\varepsilon \prod_{j=1}^m \mathscr{M}_N(f_j)(y_0),$$

where $\mathscr{M}_N$ is the grand maximal function defined as

$$\mathscr{M}_N(f)(x) = \sup_{\varphi \in \mathscr{F}_N} \sup_{\substack{y \in \mathbf{R}^n \\ |y-x| \leq t}} |(\varphi_t * f)(y)|,$$

where

$$\mathscr{F}_N = \left\{ \varphi \in \mathscr{S}(\mathbf{R}^n) :\ \mathfrak{N}_N(\varphi) \leq 1 \right\},$$

$$\mathfrak{N}_N(\varphi) = \int_{\mathbf{R}^n} (1+|x|)^N \sum_{|\alpha| \leq N+1} |\partial^\alpha \varphi(x)|\, dx,$$

and $N = \max_j [\frac{n}{p_j}] + 1$.

Then term I tends to 0 once we observe that $\|\sum_{i_j=L}^\infty \lambda_{j,i_j} a_{j,i_j}\|_{H^{p_j}} \to 0$ as $L \to \infty$ and term I can be controlled by a sum of terms with each term of the form $(\prod_{t\neq j} \|f_t\|_{H^{p_t}} (\sum_{i_j \geq L} |\lambda_{j,i_j}|^{p_j})^{1/p_j})^p$. Since δ is arbitrary, we establish that

$$T^{(\varepsilon)}(f_1, \dots, f_m) = \sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T^{(\varepsilon)}(a_{1,i_1}, \dots, a_{m,i_m}) \text{ a.e..} \tag{8}$$

To remove the ε in the preceding equality, we claim that

$$\begin{aligned} &\sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T^{(\varepsilon_k)}(a_{1,i_1}, \dots, a_{m,i_m}) \to \\ &\qquad \sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T(a_{1,i_1}, \dots, a_{m,i_m}) \end{aligned} \tag{9}$$

in measure as $\varepsilon_k \to 0$. Hence once we fix the f_j's and their atomic decomposition $f_j = \sum_j \lambda_{j,i_j} a_{j,i_j}$ for $1 \leq j \leq m$, we can find a subsequence $\varepsilon_{k_l} \to 0$ such that

$$\begin{aligned} &\sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T^{(\varepsilon_{k_l})}(a_{1,i_1}, \dots, a_{m,i_m})(x) \to \\ &\qquad \sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} T(a_{1,i_1}, \dots, a_{m,i_m})(x) \text{ a.e.} \end{aligned} \tag{10}$$

Combining all these results and (8) we can get the desired equality.

Now let us prove the claimed convergence in measure (9). We want to estimate

$$\left| \left\{ \left| \sum_{i_1} \cdots \sum_{i_m} \lambda_{1,i_1} \cdots \lambda_{m,i_m} (T^{(\varepsilon)}(a_{1,i_1}, \dots, a_{m,i_m}) - T(a_{1,i_1}, \dots, a_{m,i_m})) \right| > \delta \right\} \right|$$

$$\leq \Big|\{|\sum_{\max(i_1,\dots,i_m)\leq L} \lambda_{1,i_1}\cdots\lambda_{m,i_m}(T^{(\varepsilon)}-T)(a_{1,i_1},\dots,a_{m,i_m})|>\delta/2\}\Big|$$
$$+\Big|\{\big|\sum_{\max(i_1,\dots,i_m)\geq L+1} \lambda_{1,i_1}\cdots\lambda_{m,i_m}(T^{(\varepsilon)}-T)(a_{1,i_1},\dots,a_{m,i_m})\big|>\delta/2\}\Big|.$$

We bound the second term by $C(2/\delta)^p\sum_{\max(i_1,\dots,i_m)\geq L+1}|\lambda_{1,i_1}\cdots\lambda_{m,i_m}|^p$, which turns out to be less than a given $\tau>0$ if L is large. Once we fix L, the first term can be controlled by τ too for ε_k small since $T^{(\varepsilon_k)}\to T$ in L^q. Therefore the claimed convergence is valid. □

5. Proof of Lemma 4.2

Now we will prove that if $f_j\in L^2\cap H^{p_j}$, then

$$\left|\int_{\mathbf{R}^{mn}} K^{(\varepsilon)}(y_0,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m\right|\leq C_\varepsilon\prod_{j=1}^m \mathscr{M}_N(f_j)(y_0)\,,$$

where $\mathscr{M}_N$ is the grand maximal function.

We will use the following fact: Let F be a $\mathscr{C}^\infty$ function on $\mathbf{R}^n$ supported in $[-A/2,A/2]^n$ for some $A>0$. Then we have

$$F(x)=\frac{1}{A^n}\sum_{k\in\mathbf{Z}^n}\widehat{F}(k/A)e^{2\pi ik\cdot x/A}\chi_{[-A/2,A/2]^n}(x)\,.$$

(This is proved via a Fourier series expansion of the function $F(Ax)$ on the cube $[-1/2,1/2]^n$.)

For every $x\in\mathbf{R}^n$ define a function $K^{(\varepsilon,x)}$ on $(\mathbf{R}^n)^m$ via

$$K^{(\varepsilon,x)}(t_1,\dots,t_m)=K^{(\varepsilon)}(x,x+t_1,\dots,x+t_m)$$

Then we have

$$K^{(\varepsilon,x)}(t_1,\dots,t_m)=K^{(\varepsilon)}(x,y_1,\dots,y_m)$$

when $t_j=y_j-x$. The function

$$(t_1,\dots,t_m)\mapsto K^{(\varepsilon,x)}(t_1,\dots,t_m)$$

is supported in the ball $B(0,2/\varepsilon)^m$ which is contained in $[-2/\varepsilon,2/\varepsilon]^{nm}$. We expand in Fourier series $K^{(\varepsilon,x)}$ on the cube $[-4/\varepsilon,4/\varepsilon]^{nm}$ to obtain

$$\begin{aligned}&K^{(\varepsilon,x)}(t_1,\dots,t_m)\\&\quad=\Big(\frac{\varepsilon}{8}\Big)^{mn}\sum_{\vec{k}\in(\mathbf{Z}^n)^m}C_{\vec{k}}(x,\varepsilon)e^{\frac{2\pi\varepsilon}{8}i(k_1\cdot t_1+\cdots+k_m\cdot t_m)}\Theta\big(|t_1|\varepsilon\big)\cdots\Theta\big(|t_m|\varepsilon\big)\,,\end{aligned}$$

where $(t_1,\dots,t_m)\mapsto \Theta\big(|t_1|\varepsilon\big)\cdots\Theta\big(|t_m|\varepsilon\big)$ is a smooth function supported in $B(0,4/\varepsilon)^m$ and is equal to 1 on $B(0,2/\varepsilon)^m$, which contains the support of $K^{(\varepsilon,x)}$. Also,

$$C_{\vec{k}}(x,\varepsilon)=\int K^{(\varepsilon,x)}(t_1,\dots,t_m)e^{-\frac{2\pi\varepsilon}{8}i(k_1\cdot t_1+\cdots+k_m\cdot t_m)}dt_1\cdots dt_m\,.$$

To estimate $C_{\vec{k}}(x,\varepsilon)$ we integrate by parts with respect to the differential operator $(I-\Delta_{t_1})^M\cdots(I-\Delta_{t_m})^M$. We note that the hypothesis (1) on K (which is also valid for $K^{(\varepsilon)}$) implies that

$$|\partial^\alpha K^{(\varepsilon,x)}(t_1,\dots,t_m)|\le A'_\alpha\,(|t_1|+\cdots+|t_m|)^{-mn-|\alpha|}$$

uniformly in x for all $|\alpha|\le I$. Integration by parts gives that

$$|C_{\vec{k}}(x,\varepsilon)|\le C_M^\varepsilon\,(1+|k_1|^2)^{-M}\cdots(1+|k_m|^2)^{-M}$$

for any $M>0$ such that $2mM\le I$.

Then we can write

$$\int_{\mathbf{R}^{mn}}K^{(\varepsilon)}(y_0,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m$$

as a sum

$$\sum_{\vec{k}\in(\mathbf{Z}^n)^m}C_{\vec{k}}(x,\varepsilon)\prod_{j=1}^m\int e^{\frac{2\pi\varepsilon}{8}ik_j\cdot(x-y_j)}\Theta\big(|x-y_j|\varepsilon\big)f_j(y_j)\,dy_j.$$

All the functions inside the integral are multiples of normalized bumps whose $\mathfrak{N}_N$ norm is at most a multiple of $(1+|k_j|)^{N+1}$. Taking $2M>N+1+n$ we obtain the required conclusion in view of the decay of the sum in $\vec{k}$. Note that we need here $I=m(N+1+n)$, where I is as in Definition 1.1.

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